Configurations

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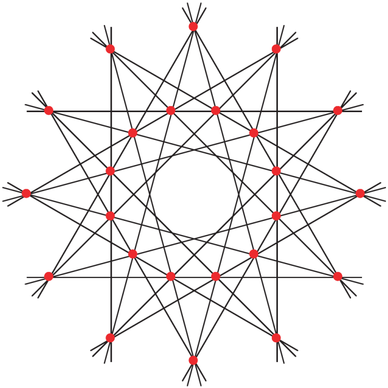
Outline

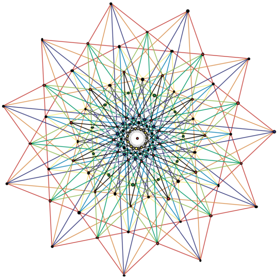
Introduction

What is a configuration?

k-configuration, specifically an (n_k) configuration is a set of n points and n lines such that every line is incident with k points (and vice-versa).

Configurations



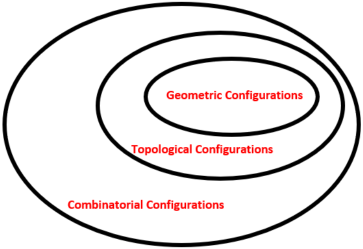


Basic configuration theory

Geometric Configuration: an image of points and lines in the (extended) Euclidean plane.

Topological Configuration: similar to a geometric configuration, except that the "lines" are pseudo-lines.

Combinatorial (set) Configuration: an abstract representation listing the incidences of points for each line.



Properties and Relationships

Two configurations are **isomorphic** if they admit labels such that there exists a 1-to-1 mapping from points to points (and lines to lines) that preserves incidences. Such a map is called an **automorphism**.

Remark: There can be infinitely many geometric (or topological) realizations of a set configuration and we say that a set configuration **underlies** a given geometric (or topological) configuration.

Examples of Configurations

1	1	1	2	2	2	3	3	3
4	5	6	4	5	6	4	5	6
7	8	9	8	9	7	9	7	8

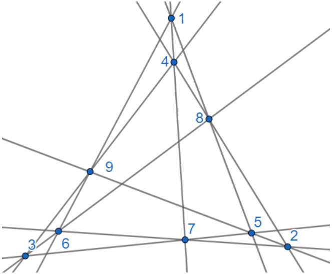


Figure 1: The $(9_3)_1$ configuration and its underlying set

Examples of Isomorphisms

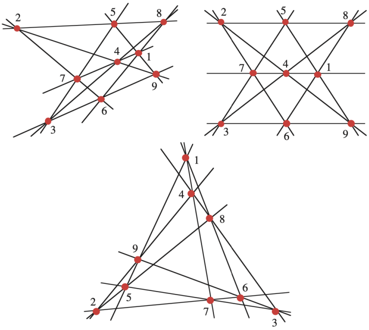


Figure 2: Three isomorphic realizations of the $(9_3)_1$ configuration

State of the Field

Our understanding of geometric configurations is largely limited to various constructions. Not many general results exist in proving the existence of geometric configurations for arbitrary n or k.

Enumerations of 3-configurations

Table 1: Number of non-isomorphic configurations

n	$\#_c(n)$	$\#_t(n)$	$\#_g(n)$
≤ 6	0	0	0
7	1	0	0
8	1	0	0
9	3	3	3
10	10	10	9
11	31	31	31
12	229	229	229
13	$2,036$		
14	$21,399$		
15	$245,342$		
16	$3,004,881$		
17	$38,904,499$		
18	$530,452,205$		
19	$7,640,941,062$		

Outline

Group Structure of Configurations

Motivation

If a given set configuration exhibits some automorphic structure, when do geometric realizations exist that exhibit that same structure as symmetries?

Geometric Symmetries

For a given geometric configuration, we call transformations that map points to points (and lines to lines) that preserves incidences a **symmetry**. The set of all these symmetries form a group which acts on the set of labels of a configuration.

Geometric configurations can exhibit dihedral or cyclic symmetries (or both)

Geometric Symmetry Example Cyclic

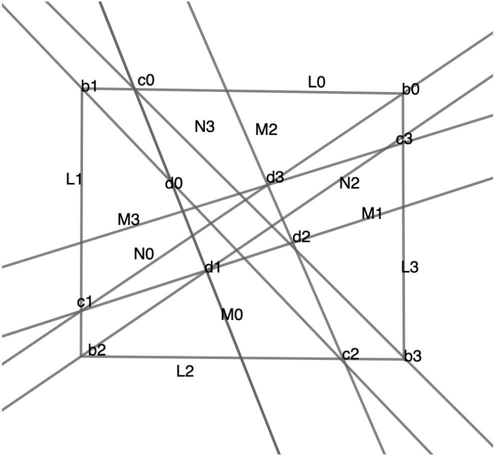


Figure 3: Symmetry group c_4

Geometric Symmetry Example Dihedral

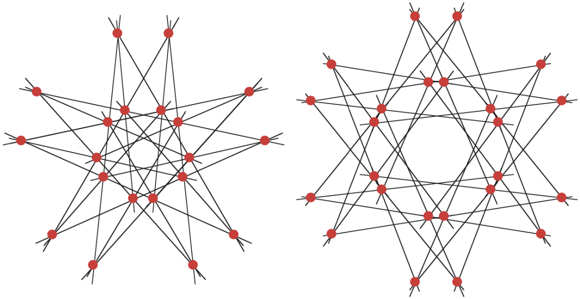


Figure 4: Configurations with dihedral symmetries

Automorphism groups

If we concern ourselves instead with set-configurations, we refer to a point to point transformation that preserve incidences as an **automorphism**. The set of all automorphisms form a group...

Automorphism group as a subgroup

Theorem

For a configuration C with n lines and k points, its automorphism group ($\mathrm{Aut}(C)$) is a subgroup if $S_n \times S_k$.

As customary, we want to show that $\mathrm{Aut}(C) \in S_n \times S_k$, contains inverses and identities.

Subgroup Proof (1)

We can represent any automorphism as a permutation of lines σ_n paired with a permutation of points σ_k. And so, we have containment.

Subgroup Proof (2)

Given an automorphism (σ_n, σ_k), we want to show that $(\sigma_n, \sigma_k)^{-1}$ is also an automorphism (i.e. it preserves incidences). We notice that permutations are cyclic.
Let o_n be the order of σ_n and o_k be the order of σ_k. Then,

$$\begin{aligned}(\sigma_n^{o_n o_k}, \sigma_k^{o_n o_k}) = e \implies (\sigma_n^{o_n o_k - 1}, \sigma_k^{o_n o_k - 1}) &= ((\sigma_n^{o_n o_k})^{-1}, (\sigma_k^{o_n o_k})^{-1}) \\ &= (\sigma_n^{o_n o_k}, \sigma_k^{o_n o_k})^{-1}\end{aligned}$$

Since $(\sigma_n^{o_n o_k - 1}, \sigma_k^{o_n o_k - 1})$ is just the composition of $o_n o_k - 1$ automorphic transformations, it must also be an automorphism. Thus $(\sigma_n, \sigma_k)^{-1} \in \mathrm{Aut}(C)$

Subgroup Proof (3)

The identity automorphism preserves incidences by definition and so we indeed have a subgroup. $\square$

Classification of automorphism groups

Given a set-configuration, determining its automorphism group is non-trivial. We rely on some tools in graph theory to help us out.

The Levi Graph

The **Levi graph** is a bipartite graph whose with a black vertex for each point in a configuration and a white vertex for each line in the configuration. An edge exists between a black and white vertices if and only if a given the respective point is incident to the respective line in the configuration.

Every set configuration admits a unique Levi graph and thus, by understanding the *graph* automorphisms of a Levi graph, we can understand the automorphisms of the configuration it represents.

Levi Graph Example 1

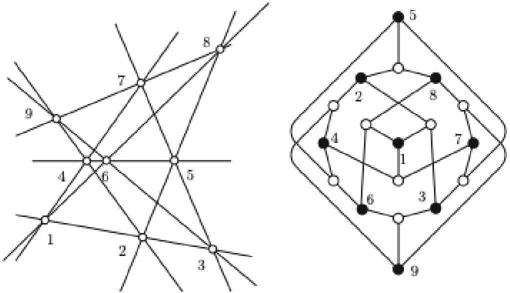


Figure 5: The Pappus configuration $(9_3)_1$. On the left is the geometric configuration, and on the right is its corresponding Levi graph.

Levi Graph Example 2

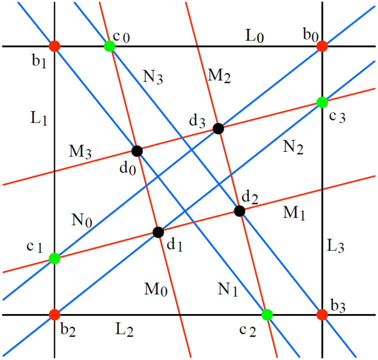


Figure 6: The geometric configuration (12_3)

Levi Graph Example 2

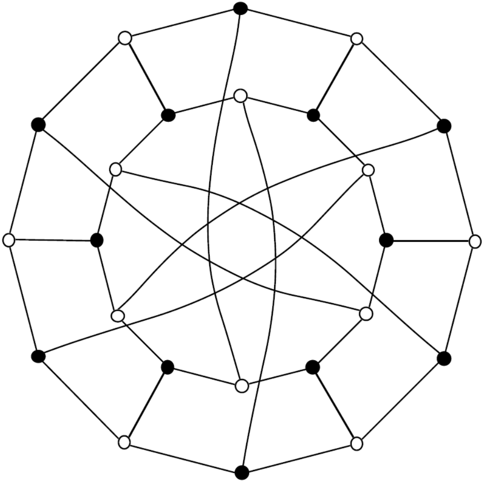


Figure 7: The Levi graph of a (12_3) configuration

Using Levi Graphs

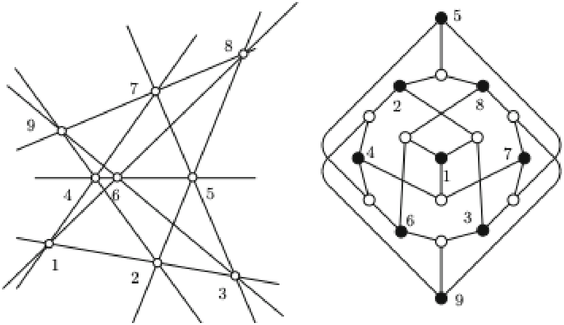
Every set-configuration admits a unique Levi graph.

A color and incidence preserving automorphism of the Levi graph is then an automorphism of the set-configuration it represents.

Conjectures and Positive Examples

To investigate our posed question we narrow our gaze to cyclic subgroups. We examine an example when a cyclic automorphism subgroup implies a geometric ones. We proceed to offer some general conjectures.

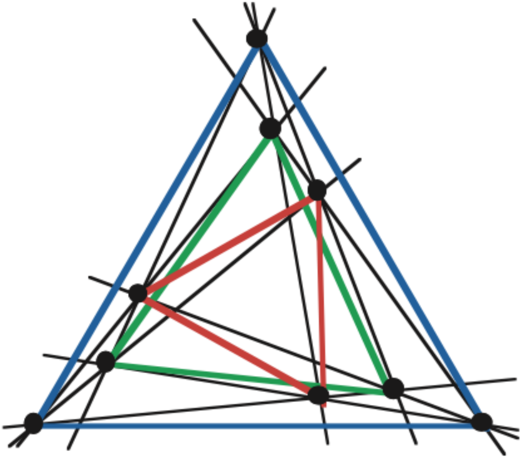
The Pappus Automorphisms



The permutation of vertices (123)(456)(789) is an automorphism of the Levi graph. Furthermore this permutation has order three and thus generates a subgroup isomorphic to $\mathbb{Z}/3\mathbb{Z}$. We might suspect that there exist another realization of the Pappus configuration with 3-fold symmetry.

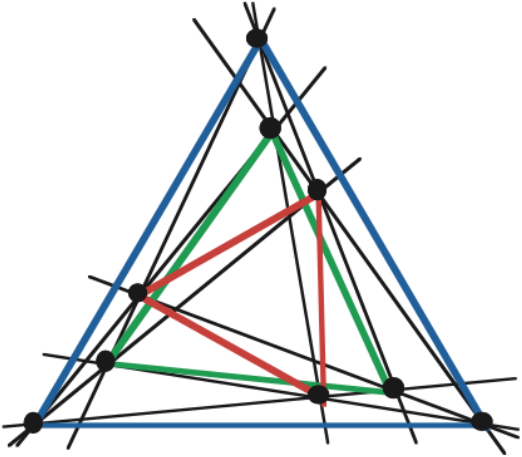
The Pappus Automorphisms

And one does indeed exist. Note this is isomorphic to the previous.



Orbits

We can color code orbits of points in the laterals that connect them.



Symmetries and Automorphisms: Conjectures

Let C be a set-configuration of k lines and points.
Conjecture: If the automorphism group of C contains a cyclic subgroup of order t with $t < k$ then $t \mid k$ and there exists a geometric realization of C with t-fold rotational symmetry.

The Pappus configuration we have discussed is a positive example.

Symmetries and Automorphisms: Conjectures (2)

Let C be a set-configuration of k lines and points.
Conjecture: Let p prime with $p < k$. If $p \mid \#\mathrm{Aut}(C)$ than there exist a geometric realization of C with p-fold rotational symmetry and $p \mid k$.

Symmetries and Automorphisms: Conjectures (2) contd.

By Sylow 1, if $p \mid \#\mathrm{Aut}(C)$ then there must exist a cyclic subgroup of order p. Then the Levi graph of C must also have a cyclic subgroup of order p. If we assume the above conjecture, we are done.

Conclusion

By definition, the study of automorphism groups of configurations is equivalent to the study of automorphisms of bipartite graphs. General results on the existence of their geometric realizations seem quite intractable given our (humanities) current understanding. As highlighted earlier, we only know of the existence of a select few by construction.

Generalized Cyclic Configurations

Outline

What is a Generalized Cyclic Configuration?

Definition

A Generalized Cyclic Configuration is one that is underlied by the table shown below. This table is denoted as $\mathcal{J}(n, a, b)$.

1	2	3	4	...	n-3	n-2	n-1	n
1+a	2+a	3+a	4+a	...	n-3 + a	n-2 + a	n-1+a	n+a
1+b	2+b	3+b	4+b	...	n-3 + b	n-2 + b	n-1+b	n+b

Outline

Existence Statements

Theorem (1)

Combinatorial configurations (n_3) *exist if and only if* $n \geq 7$

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Topological configurations (n_3) *exist if and only if* $n \geq 9$

Existence Statements

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Theorem (2)

Topological configurations (n_3) *exist if and only if* $n \geq 9$

Theorem (3)

Geometric configurations (n_3) *exist if and only if* $n \geq 9$

Classical Approach

Usually, Theorems 1 and 3 are proved using $\mathcal{J}(n,1,3)$.

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The question then arises:

For what other values a, b can the table $\mathcal{J}(n, a, b)$ be used to prove these theorems?

Combinatorial Configurations using $\mathcal{J}(n, a, b)$

Theorem (1)

For a fixed n and the generalized cyclic table $C = \mathcal{J}(n, a, b)$, $a < b$, if $b \notin \{n - a, \frac{n+a}{2}, \frac{n}{2} + a, \frac{n}{2}, 2a\}$ and $a \neq \frac{n}{2}$, then C is a combinatorial configuration.

Combinatorial Configurations using $\mathcal{J}(n, a, b)$

Outline of proof:

Since $a < b$, C is not a combinatorial configuration when there are at least two columns $A = \{j, j + a, j + b\}, B = \{k, k + a, k + b\}$, $1 \leq j, k \leq n$, such that $|A \cap B| \geq 2$.

[1] with everything mod n

Combinatorial Configurations using $\mathcal{J}(n, a, b)$

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$|A \cap B| \geq 2$ when one of the following is true[1]:

1) $j = k + a$ and $(j + a = k$ or $j + a = k + b)$

2) $j + a = k + b$ and $(j + b = k + a$ or $j + b = k)$

3) $j + b = k$ and $(j = k + b$ or $j = k + a)$

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2) $j + a = k + b$ and ($j + b = k + a$ or $j + b = k$)

3) $j + b = k$ and ($j = k + b$ or $j = k + a$)

The above can be simplified to get $b \in \{n - a, \frac{n+a}{2}, \frac{n}{2} + a, \frac{n}{2}, 2a\}$ or $a = \frac{n}{2}$ $\square$

[1] with everything mod n

Combinatorial Configurations using $\mathcal{J}(n, a, b)$

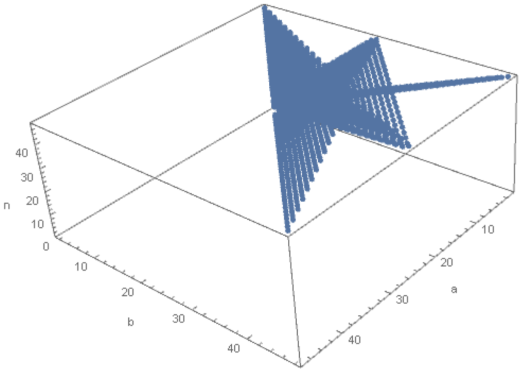

Figure 8: Values of a, b such that the table $\mathcal{J}(n, a, b)$ is not a valid configuration for $8 \leq n < 50$

Geometric Configurations using $\mathcal{J}(n, a, b)$

Definition

If a combinatorial configuration contains a Hamiltonian multilateral, then there exists a sequence $P_0, L_0, P_1, \ldots P_{n-1}, L_{n-1}, P_n$ such that each L_i is incident with P_i and P_{i+1}. Moreover, this sequence contains all points and lines once, and only once.

Notation:
A generalized cyclic table that contains a Hamiltonian multilateral using rows q and r will be said to contain an $H_{q,r}$ path.

1	2	3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9	10	1
4	5	6	7	8	9	10	1	2	3

Figure 9: $\mathcal{J}(10, 1, 3)$ contains both an $H_{1,2}$ (shown) and a $H_{1,3}$ path (not shown).

Geometric Configurations using $\mathcal{J}(n, a, b)$

Notice that the existence of an $H_{q,r}$ path for $\mathcal{J}(n, a, b)$ can be easily shown, and can be explicitly defined[2]:

> $H_{1,2}$ exists if $\gcd(n, a) = 1$. If it does,
> $H_{1,2} = \{j + ia | 1 \leq i < n\}$

[2] with everything mod n

Geometric Configurations using $\mathcal{J}(n, a, b)$

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$H_{1,2}$ exists if $\gcd(n, a) = 1$. If it does,
$H_{1,2} = \{j + ia | 1 \leq i < n\}$

$H_{1,3}$ exists if $\gcd(n, b) = 1$. If it does,
$H_{1,3} = \{j + ib | 1 \leq i < n\}$

[2] with everything mod n

Geometric Configurations using $\mathcal{J}(n, a, b)$

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$H_{1,3}$ exists if $\gcd(n, b) = 1$. If it does,
$H_{1,3} = \{j + ib | 1 \leq i < n\}$

$H_{2,3}$ exists if $\gcd(n, b - a) = 1$. If it does,
$H_{2,3} = \{j + a + i(b - a) | 1 \leq i < n\}$

[2] with everything mod n

Theorem (2)

For $n \geq 9$, a generalized cyclic configuration $C = \mathcal{J}(n, a, b), a < b$ can be geometrically realized following an algorithm described by Grünbaum if one of the following is true:
1) C contains an $H_{1,2}$ path and $b = 3a$ or
2) C contains an $H_{2,3}$ path and $b = \frac{3}{2}a$

Geometric Configurations using $\mathcal{J}(n, a, b)$

Grünbaum's algorithm:
Any $\mathcal{J}(n, 1, 3)$ combinatorial configuration, $n \geq 9$, can be geometrically realized by placing the elements 2 to $n-3$ of R_1 and R_2 (which form a Hamiltonian multilateral) in a "zig-zag" pattern.

1	2	3	4	...	n-3	n-2	n-1	n
2	3	4	5	...	n-2	n-1	n	1
4	5	6	7	...	n	1	2	3

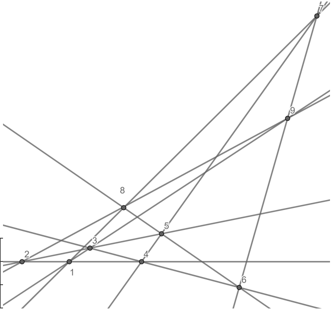
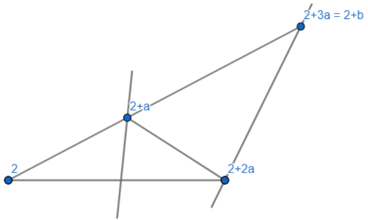


Figure 10: $\mathcal{J}(n, 1, 3)$ configuration table and geometric realization

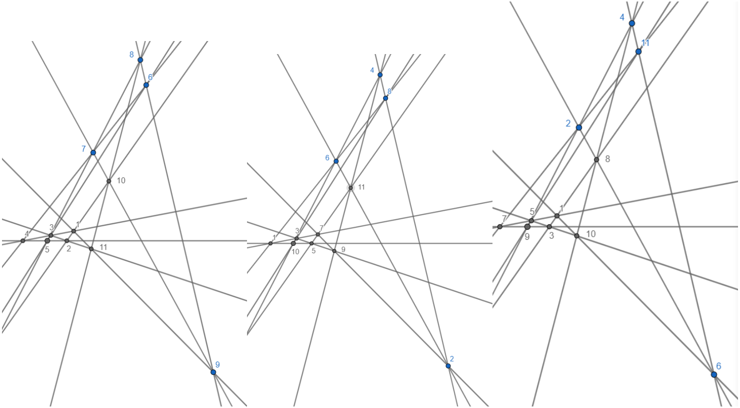
Geometric Configurations using $\mathcal{J}(n, a, b)$

Outline of proof of Theorem 2:
As the proof is nearly identical for both cases, assume that R_1 and R_2 form the Hamiltonian multilateral. In addition, let 2 be the starting point.



Making the zig-zag pattern with the path $\{2, 2+a, 2+2a, 2+3a\}$, we see that $2+3a$ must be the same point as $2+b$, indicating that $b = 3a$ $\square$

Geometric Configurations using $\mathcal{J}(n, a, b)$



(a) $\mathcal{J}(11, 2, 3)$ (b) $\mathcal{J}(11, 2, 6)$ (c) $\mathcal{J}(11, 4, 6)$

Figure 11: Examples of valid geometric configurations

Open Question

For the (valid) values of a, b such that $\mathcal{J}(n, a, b)$ cannot be geometrically realized using the previous construction, is there another way to realize these combinatorial configurations, or are they unrealizable?

Outline

What is an irreducible configuration?

Definition

A combinatorial configuration is reducible if there is a point C, collinear with points A, B, with points A', A'', and with points B', B'' such that, if C is removed, then the six remaining points can be rearranged to form $\{A, A', A''\}$ and $\{B, B', B''\}$

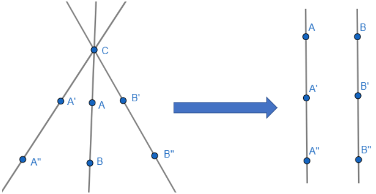


Figure 12: Visualization of what it means to be reducible.

What is an irreducible configuration?

Definition

A combinatorial configuration that cannot be reduced in this manner is called irreducible.

What is an irreducible configuration?

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A combinatorial configuration that cannot be reduced in this manner is called irreducible.

Example

1	2	3	4	5	6	7	8
2	3	4	5	6	7	8	1
4	5	6	7	8	1	2	3

Figure 13: $\mathcal{J}(8, 1, 3)$ is an irreducible configuration

Removing point 3, we see that, for example, no matter which new point we choose for the line $\{8, 1\}$, there will never be a new configuration.

When is a configuration irreducible?

Theorem (Boben)

A connected configuration C is irreducible if and only if either:

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$$C = \mathcal{J}(n, 1, 3)$$

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C *is the Pappus configuration* $(9_3)_1$

When is a configuration irreducible?

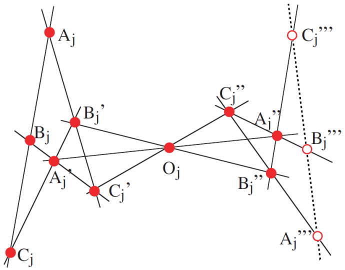
Theorem (Boben)

A connected configuration C is irreducible if and only if either:

- $C = \mathfrak{J}(n, 1, 3)$
- *C is the Pappus configuration $(9_3)_1$*
- $n = 10m, m \geq 1$, *and* $C = \mathbb{M}(m), \mathbb{M}^*(m)$, *or* $\mathbb{M}^{**}(m)$.

$\mathbb{M}(m)$

$\mathbb{M}(m), \mathbb{M}^*(m), \mathbb{M}^{**}(m)$ are configurations made up of "modules", each consisting of 10 points, which can be drawn in the following way:



$\mathbb{M}(m)$

In each case, for $1 \leq j < n$, A_j''' is the same point as A_{j+1}, B_j''' is the same point as B_{j+1}, and C_j''' is the same point as C_{j+1}.

When $j = n$:
$\mathbb{M}(m)$ maps A_n''' to A_1, B_n''' to B_1, and C_n''' to C_1.
$\mathbb{M}^*(m)$ maps A_n''' to C_1, B_n''' to B_1, and C_n''' to A_1.
$\mathbb{M}^{**}(m)$ maps A_n''' to C_1, B_n''' to A_1, and C_n''' to B_1

Example: $\mathcal{J}(n, 1, 4)$

From the Irreducibility Theorem, it appears that $C = \mathcal{J}(n, 1, 4)$, $n \geq 9$, is reducible given $a = 1$ and $b = 4$

Example: $\mathcal{J}(n, 1, 4)$

We have seen that C is only reducible for $n \geq 12$

1	2	3	4	5	6	7	8	9	10	11	12
2	3	4	5	6	7	8	9	10	11	12	1
5	6	7	8	9	10	11	12	1	2	3	4

Figure 14: $\mathcal{J}(12, 1, 4)$ is an reducible configuration

Example: $\mathcal{J}(n, 1, 4)$

For when $n = 9$, using the prior definitions, we can show that C is irreducible. This may be due to the cyclic configuration being isomorphic to the Pappus configuration.
However, this configuration is not isomorphic to the Pappus configuration, which gives an immediate contradiction to the irreducibility theorem.

6	7	8	9	1	2	3	4	5
1	2	3	4	5	6	7	8	9
2	3	4	5	6	7	8	9	1
5	6	7	8	9	1	2	3	4

Figure 15: $\mathcal{J}(9, 1, 4)$ is irreducible

Example: $\mathcal{J}(n, 1, 4)$

For when $n = 10$, C is irreducible. However, we note that this configuration is not isomorphic to the $\mathbb{M}(m)$, $\mathbb{M}^*(m)$, and $\mathbb{M}^{**}(m)$ configurations, which gives an immediate contradiction to the irreducibility theorem.

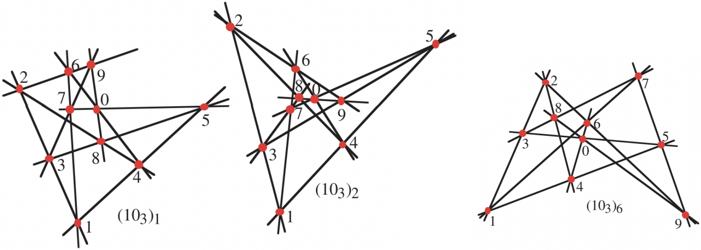


Figure 16: The three Martinetti configurations are irreducible

Example: $\mathcal{J}(n, 1, 4)$

For when $n = 11$, C is irreducible. We note that this is still an area that needs further investigation.

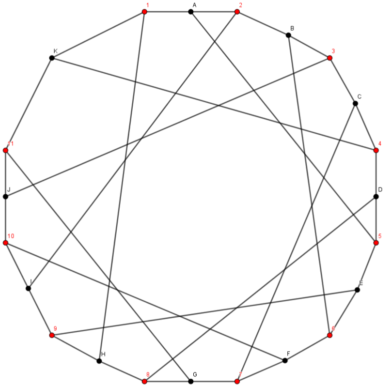


Figure 17: Levi graph for the $\mathcal{J}(11, 1, 4)$ configuration

Example: $\mathcal{J}(n, 1, 4)$

Future work:

Can a point be added to $\mathbb{M}(1)$, $\mathbb{M}^*(1)$, or $\mathbb{M}^{**}(1)$, that gives rise to the configuration $\mathcal{J}(11, 1, 4)$ such that $\mathcal{J}(11, 1, 4)$ is shown to be reducible?

Example: $\mathcal{J}(n, 1, 4)$

Future work:

- Can a point be added to $\mathbb{M}(1)$, $\mathbb{M}^*(1)$, or $\mathbb{M}^{**}(1)$, that gives rise to the configuration $\mathcal{J}(11, 1, 4)$ such that $\mathcal{J}(11, 1, 4)$ is shown to be reducible?
- How does the Levi graph differ between an irreducible $((n-1)_3)$ configuration and a reducible (n_3) configuration?

Outline

Miscellaneous

Recall Theorem 1:
For a fixed n and the generalized cyclic table $C = \mathcal{J}(n, a, b), a < b$, if $b \notin \{n - a, \frac{n+a}{2}, \frac{n}{2} + a, \frac{n}{2}, 2a\}$ and $a \neq \frac{n}{2}$, then C is a combinatorial configuration.

Put differently, for a fixed n, if (a, b) is a point on the line $n = 2b - 2a, n = 2a, n = 2b, n = a + b, n = 2b - a,$ or $2a = b$, then $C' = \mathcal{J}(n, a, b)$ is not a configuration.

Miscellaneous

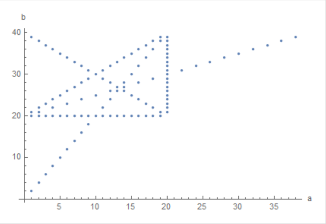
Theorem

If $C' = \mathfrak{J}(n, a, b)$, $b > a$ is not a combinatorial configuration, and $6|n$, then the centroid of the triangle bounded by the lines $n = 2b - 2a, n = 2a, n = 2b$ are the values a, b such that C' contains two columns A, B, $A \neq B$[3], where $|A \cap B| = 3$.

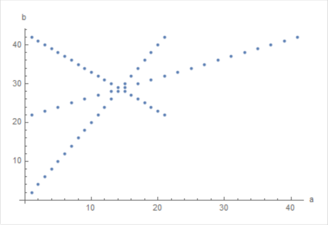
[3] Equality here is defined as follows: if $A[i]$ is the i-th element of line A, then $A = B \iff \forall n, A[n] = B[n]$

Miscellaneous

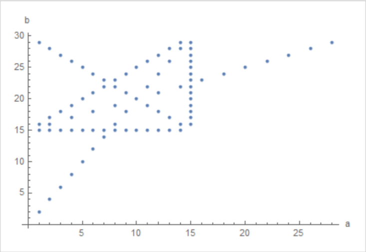
Values a, b for which $\mathcal{J}(n, a, b)$ is not a combinatorial configuration:



(a) $\mathcal{J}(40, a, b)$



(b) $\mathcal{J}(43, a, b)$



(a) $\mathcal{J}(30, a, b)$

Miscellaneous

Outline of Proof:

Show that the centroid of the triangle is at $(\frac{n}{3}, \frac{2n}{3})$.

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Lemma

Let A, B both be columns of $\mathfrak{J}(n, a, b)$, $a < b$, such that $A \neq B$. Then, $\forall A$, $\exists B$ s.t. $|A \cap B| = 3 \iff 3|n$ and $a = \frac{n}{3}, b = \frac{2n}{3}$.

References

Branko Grunbaum. *Configurations of Lines and Spaces*. American Mathematical Society, 2009.